MA 3046 - Matrix Analysis
The Notion of Generalized Coordinates

Initially a vector is a physical, geometrical object with magnitude and direction:

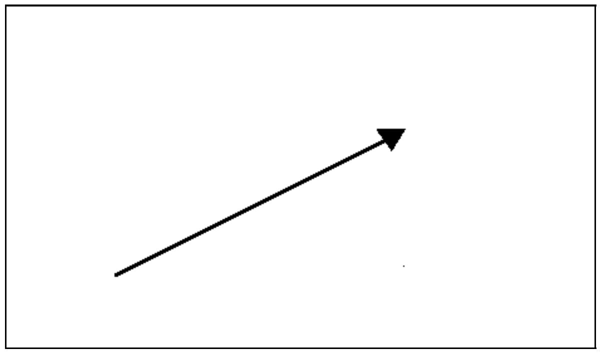

Figure 1

until we lay down a coordinate system, e.g:

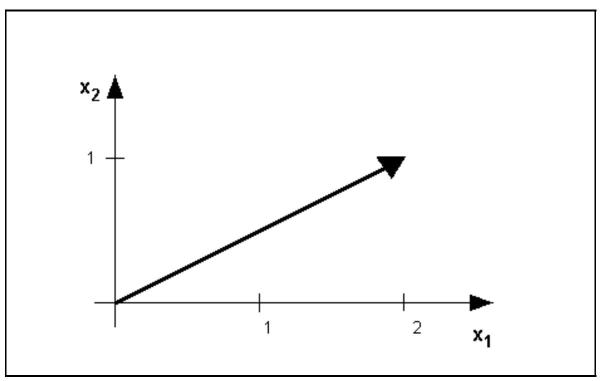

Figure 2

in which case, the vector can now be represented by its coordinates with respect to that system, which, in this case are

$$\mathbf{x} = [2, 1]$$

(Note that what these coordinates mean, operationally, is that we can find the tip of the vector by first moving two units in the x_1 direction, then followed by a move of one unit in the x_2 direction.)

However, what says that the coordinate system shown in Figure 2 is the only choice. Why, for example, could not another observer, viewing the vector from a different perspec-

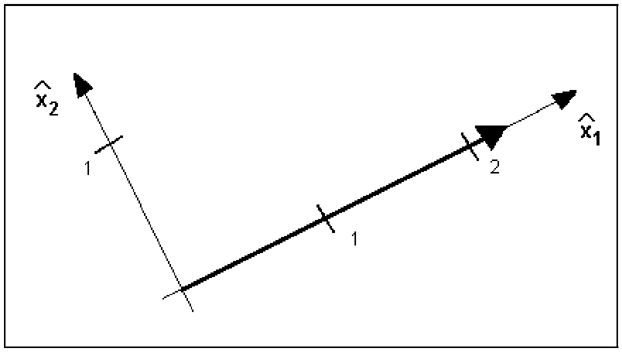

Figure 3

tive, use the coordinate system shown in Figure 3? After all, it's still the same physical vector as in Figure 2.

But to reach the tip of the vector in this new system, we must now first move $\sqrt{5}$ units in the $\hat{x}_1$ direction, then followed by a move of zero units in the $\hat{x}_2$ direction. In other words, in this new system, the coordinates of the vector are

$$\hat{\mathbf{x}} = [\sqrt{5}, 0]$$

In other words, what this simple example shows is it is impossible to separate the coordinates of a vector from the coordinate system in which that vector is constructed!

MA 3046 - Matrix Analysis
Coordinates in Terms of Nonstandard Bases

By definition, a *basis* is any linearly independent spanning set.

An *ordered basis* is a basis, each of whose elements carries an explicit or implicit numerical "tag," so we can immediately say which is the first, the fifth, etc., i.e.

$$\mathcal{B} = \left\{ \mathbf{b}^{(1)} \ , \ \mathbf{b}^{(2)} \ , \ \cdots \ , \ \mathbf{b}^{(n)} \right\}$$

Ordered bases are nothing new. For example, $\Re^3$ implicitly uses the ordered basis,

$$\mathbf{e}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{e}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \mathbf{e}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(the so-called standard basis for $\Re^3$), and so, we know immediately from

$$\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \quad \text{that} \quad x = 3 \ , \quad y = -1 \ , \quad \text{and} \quad z = 4$$

(Imagine the confusion, for example, in air traffic control, if, in some countries, the second component, not the third, represented altitude!)

But if $\mathcal{B}$ is any ordered basis, we know that we can *uniquely* represent any given vector $\mathbf{x}$ as

$$\mathbf{x} = \alpha_1 \mathbf{b}^{(1)} \ + \ \alpha_2 \mathbf{b}^{(2)} \ + \ \cdots \ + \ \alpha_n \mathbf{b}^{(n)} \tag{1}$$

We call the α_i the coordinates of $\mathbf{x}$ relative to the ordered basis $\mathcal{B}$, and symbolize them by $[\mathbf{x}]_{\mathcal{B}}$. The coordinates in terms of a general ordered basis are interpreted in $\Re^m$ very much as we interpret the coordinates in terms of the standard basis, i.e. if

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

then we simply:

(i) First move $\alpha_1 \, \| \, \mathbf{b}^{(1)} \, \|$ in the direction of $\mathbf{b}^{(1)}$, then

(ii) Move $\alpha_2 \, \| \, \mathbf{b}^{(2)} \, \|$ in the direction of $\mathbf{b}^{(2)}$, then ...

Moreover, if the $\mathbf{b}^{(i)}$ are vectors from $\Re^m$, then (1) becomes, in block matrix form:

$$\underbrace{\begin{bmatrix} \mathbf{b}^{(1)} & \vdots & \mathbf{b}^{(2)} & \vdots & \ldots & \vdots & \mathbf{b}^{(n)} \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{x}$$

Based on this, we have, in $\Re^m$, the fundamental relationship

$$\mathbf{B}\,[\mathbf{x}]_{\mathcal{B}} \;=\; x \tag{2}$$

where $\mathbf{B}$ is the matrix whose columns are simply the elements of the ordered basis $\mathcal{B}$, in order. In light of this, we then see that whenever we consider the questions of existence and uniqueness of solutions to the system of linear equations, e.g.

$$\mathbf{A}\,\mathbf{x} \;=\; \mathbf{b}$$

all we are really tying to do is find whether or not the columns of $\mathbf{A}$ are a basis for some subspace of $\Re^m$, and, if so, whether it is possible to find the coordinates for $\mathbf{b}$ in terms of them.

Lastly, in discussing the coordinates in terms of any non-standard basis, it is important to keep in mind that all we are doing is representating the **same physical point** in terms of a different frame of reference. Therefore, if we happen to have two different ordered bases, say $\mathcal{B}$ and

$$\mathcal{C} = \left\{ \mathbf{c}^{(1)} \;,\; \mathbf{c}^{(2)} \;,\; \cdots \;,\; \mathbf{c}^{(n)} \right\}$$

for the same subspace, then

$$\mathbf{B}\,[\mathbf{x}]_{\mathcal{B}} \;=\; \mathbf{C}\,[\mathbf{x}]_{\mathcal{C}} \;=\; x$$

where $\mathbf{C}$ is the matrix whose columns are the individual vectors in the basis $\mathcal{C}$, in order. Based on this equation, it is trivial, at least in theory, to move between different coordinates for the same (sub)space, since

$$[\mathbf{x}]_{\mathcal{C}} \;=\; \mathbf{C}^{-1}\,\mathbf{B}\,[\mathbf{x}]_{\mathcal{B}} \tag{3}$$

In other texts, the matrix $\mathbf{C}^{-1}\,\mathbf{B}$ is commonly called the *transition matrix* from $\mathcal{B}$ to $\mathcal{C}$, and may be symbolized by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathbf{P}}$.